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Integral representations for solutions of some BVPs for the Lamé system in multiply connected domains

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Abstract

The present paper is concerned with an indirect method to solve the Dirichlet and the traction problems for Lamé system in a multiply connected bounded domain of $\mathbb{R}^n$, $n \geq 2$. It hinges on the theory of reducible operators and on the theory of differential forms. Differently from the more usual approach, the solutions are sought in the form of a simple layer potential for the Dirichlet problem and a double layer potential for the traction problem.

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1 Introduction

In this paper we consider the Dirichlet and the traction problems for the linearized n-dimensional elastostatics. The classical indirect methods for solving them consist in looking for the solution in the form of a double layer potential and a simple layer potential respectively. It is well-known that, if the boundary is sufficiently smooth, in both cases we are led to a singular integral system which can be reduced to a Fredholm one (see, e.g., [1]).

Recently this approach was considered in multiply connected domains for several partial differential equations (see, e.g., [2-7]).

However these are not the only integral representations that are of importance. Another one consists in looking for the solution of the Dirichlet problem in the form of a simple layer potential. This approach leads to an integral equation of the first kind on the boundary which can be treated in different ways. For $n = 2$ and Ω simply connected see [8]. A method hinging on the theory of reducible operators (see [9,10]) and the theory of differential forms (see, e.g., [11,12]) was introduced in [13] for the n-dimensional Laplace equation and later extended to the three-dimensional elasticity in [14]. This method can be considered as an extension of the one given by Muskhelishvili [15] in the complex plane. The double layer potential ansatz for the traction problem can be treated in a similar way, as shown in [16].

In the present paper we are going to consider these two last approaches in a multiply connected bounded domain of $\mathbb{R}^n$ ($n \geq 2$). Similar results for Laplace equation have been recently obtained in [17]. We remark that we do not require the use of pseudo-differential operators nor the use of hypersingular integrals, differently from other methods (see, e.g., [[18], Chapter 4] for the study of the Neumann problem for Laplace equation by means of a double layer potential).

After giving some notations and definitions in Section 2, we prove some preliminary results in Section 3. They concern the study of the first derivatives of a double layer potential. This leads to the construction of a reducing operator, which will be useful in the study of the integral system of the first kind arising in the Dirichlet problem.

Section 4 is devoted to the case $n = 2$, where there exist some exceptional boundaries in which we need to add a constant vector to the simple layer potential. In particular, after giving an explicit example of such boundaries, we prove that in a multiply connected domain the boundary is exceptional if, and only if, the external boundary is exceptional.

In Section 5 we find the solution of the Dirichlet problem in a multiply connected domain by means of a simple layer potential. We show how to reduce the problem to an equivalent Fredholm equation (see Remark 5.5).

Section 6 is devoted to the traction problem. It turns out that the solution of this problem does exist in the form of a double layer potential if, and only if, the given forces are balanced on each connected component of the boundary. While in a simply connected domain the solution of the traction problem can be always represented by means of a double layer potential (provided that, of course, the given forces are balanced on the boundary), this is not true in a multiply connected domain. Therefore the presence or absence of "holes" makes a difference.

We mention that lately we have applied the same method to the study of the Stokes system [19]. Moreover the results obtained for other integral representations for several partial differential equations on domains with lower regularity (see, e.g., the references of [20] for C^1 or Lipschitz boundaries and [21] for "worse" domains) lead one to hope that our approach could be extended to more general domains.

2 Notations and definitions

Throughout this paper we consider a domain (open connected set) $\Omega \subset \mathbb{R}^n$, $n \geq 2$, of the form $\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \bar{\Omega}_j$, where Ω_j ($j = 0, ..., m$) are $m + 1$ bounded domains of $\mathbb{R}^n$ with connected boundaries $\Sigma_j \in C^{1,\lambda}$ ($\lambda \in (0, 1]$) and such that $\bar{\Omega}_j \subset \Omega_0$ and $\bar{\Omega}_j \cap \bar{\Omega}_k = \emptyset, j, k = 1, \dots, m, j \neq k$. For brevity, we shall call such a domain an $(m + 1)$-*connected domain*. We denote by ν the outwards unit normal on $\Sigma = \partial\Omega$.

Let E be the partial differential operator

$$Eu = \Delta u + k\nabla \mathrm{div} u,$$

where $u = (u_1, ..., u_n)$ is a vector-valued function and $k > (n - 2)/n$ is a real constant. A fundamental solution of the operator $-E$ is given by Kelvin's matrix whose entries are

$$\Gamma_{ij}(x,y) = \begin{cases} \dfrac{1}{2\pi}\left(-\dfrac{k+2}{2(k+1)}\delta_{ij}\log|x-y| + \dfrac{k}{2(k+1)}\dfrac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right), & \text{if } n=2, \\ \dfrac{1}{\omega_n}\left(-\dfrac{k+2}{2(k+1)}\delta_{ij}\dfrac{|x-y|^{2-n}}{2-n} + \dfrac{k}{2(k+1)}\dfrac{(x_i-y_i)(x_j-y_j)}{|x-y|^n}\right), & \text{if } n \geq 3, \end{cases} \tag{1}$$

$i, j = 1, ..., n$, ω_n being the hypersurface measure of the unit sphere in $\mathbb{R}^n$.

As usual, we denote by $\mathcal{E}(u, v)$ the bilinear form defined as

$$\mathcal{E}(y, v) = 2\sigma_{ih}(u)\varepsilon_{ih}(v) = 2\sigma_{ih}(u)\varepsilon_{ih}(u),$$

where ε_{ih} (u) and σ_{ih} (u) are the linearized strain components and the stress components respectively, i.e.

$$\varepsilon_{ih}(u) = \frac{1}{2}(\partial_i u_h + \partial_h u_i), \quad \sigma_{ih}(u) = \varepsilon_{ih}(u) + \frac{k-1}{2}\delta_{ih}\varepsilon_{jj}(u).$$

Let us consider the boundary operator L^{ξ} whose components are

$$L_i^{\xi} u = (k - \xi)(\mathrm{div}\, u)\nu_i + \nu_j \partial_j u_i + \xi \nu_j \partial_i u_j, \quad i = 1, \ldots, n, \tag{2}$$

ξ being a real parameter. We remark that the operator L^1 is just the stress operator $2\sigma_{ih}\nu_h$, which we shall simply denote by L, while $L^{k/(k+2)}$ is the so-called pseudo-stress operator.

By the symbol $\mathscr{S}_n$ we denote the space of all constant skew-symmetric matrices of order n. It is well-known that the dimension of this space is $n(n - 1)/2$. From now on $a + Bx$ stands for a rigid displacement, i.e. a is a constant vector and $B \in \mathscr{S}_n$. We denote by $\mathcal{R}$ the space of all rigid displacements whose dimension is $n(n + 1)/2$. As usual $\{e_1, ..., e_n\}$ is the canonical basis for $\mathbb{R}^n$.

For any $1 < p < +\infty$ we denote by $[L^p(\Sigma)]^n$ the space of all measurable vector-valued functions $u = (u_1, ..., u_n)$ such that $|u_j|^p$ is integrable over Σ $(j = 1, ..., n)$. If h is any non-negative integer, $L_h^p(\Sigma)$ is the vector space of all differential forms of degree h defined on Σ such that their components are integrable functions belonging to $L^p(\Sigma)$ in a coordinate system of class C^1 and consequently in every coordinate system of class C^1. The space $[L_h^p(\Sigma)]^n$ is constituted by the vectors $(\nu_1, ..., \nu_n)$ such that ν_j is a differential form of $L_h^p(\Sigma)(j = 1, \ldots, n)$. $[W^{1,p}(\Sigma)]^n$ is the vector space of all measurable vector-valued functions $u = (u_1, ..., u_n)$ such that u_j belongs to the Sobolev space $W^{1,p}$ (Σ) $(j = 1, ..., n)$.

If B and B' are two Banach spaces and $S : B \to B'$ is a continuous linear operator, we say that S can be reduced on the left if there exists a continuous linear operator $S' : B' \to B$ such that $S'S = I + T$, where I stands for the identity operator of B and $T : B \to B$ is compact. Analogously, one can define an operator S reducible on the right. One of the main properties of such operators is that the equation $S\alpha = \beta$ has a solution if, and only if, $\langle \gamma, \beta \rangle = 0$ for any γ such that $S^*\gamma = 0$, S^* being the adjoint of S (for more details see, e.g., [9,10]).

We end this section by defining the spaces in which we look for the solutions of the BVPs we are going to consider.

Definition 2.1. *The vector-valued function u belongs to $\mathcal{S}^p$if, and only if, there exists $\phi \in [L^p(\Sigma)]^n$ such that u can be represented by a simple layer potential*

$$u(x) = \int_{\Sigma} \Gamma(x, y)\varphi(y)d\sigma_y, \quad x \in \Omega. \tag{3}$$

Definition 2.2. *The vector-valued function w belongs to $\mathcal{D}^p$if, and only if, there exists $\psi \in [W^{1,p}(\Sigma)]^n$ such that w can be represented by a double layer potential*

$$w(x) = \int_\Sigma [L_y\Gamma(x,y)]'\psi(y)d\sigma_y, \quad x \in \Omega, \tag{4}$$

where $[L_y\Gamma(x, y)]'$ *denotes the transposed matrix of* $L_y[\Gamma(x, y)]$.

3 Preliminary results

3.1 On the first derivatives of a double layer potential

Let us consider the boundary operator L^ξ defined by (2). Denoting by $\Gamma^j(x, y)$ the vector whose components are $\Gamma_{ij}(x, y)$, we have

$$\begin{aligned} L^\xi_{i,y}[\Gamma^j(x,y)] = -\frac{1}{\omega_n}\Bigg\{ \left[\frac{2+(1-\xi)k}{2(1+k)}\delta_{ij} + \frac{nk(\xi+1)}{2(k+1)}\frac{(y_i-x_i)(y_j-x_j)}{|y-x|^2}\right]\frac{(y_p-x_p)\nu_p(y)}{|y-x|^n} \\ +\frac{k-(2+k)\xi}{2(k+1)}\left[\frac{(y_j-x_j)\nu_i(y)-(y_i-x_i)\nu_j(y)}{|y-x|^n}\right]\Bigg\}. \end{aligned} \tag{5}$$

We recall that an immediate consequence of (5) is that, when $\xi = k/(2 + k)$ we have

$$L^{k/(2+k)}_{i,y}[\Gamma^j(x,y)] = \mathcal{O}(|x-y|^{1-n+\lambda}), \tag{6}$$

while for $\xi \neq k/(2 + k)$ the kernels $L^\xi_{i,y}[\Gamma^j(x,y)]$ have a strong singularity on Σ.

Let us denote by w^ξ the double layer potential

$$w^\xi_j(x) = \int_\Sigma u_i(y)L^\xi_{i,y}[\Gamma^j(x,y)]d\sigma_y, \quad j = 1, \ldots, n. \tag{7}$$

It is known that the first derivatives of a harmonic double layer potential with density ϕ belonging to $W^{1,p}(\Sigma)$ can be written by means of the formula proved in [[13], p. 187]

$$*d\int_\Sigma \varphi(y)\frac{\partial s(x,y)}{\partial \nu_y}d\sigma_y = d_x\int_\Sigma d\varphi(y)\wedge s_{n-2}(x,y), \quad x \in \Omega. \tag{8}$$

Here * and d denote the Hodge star operator and the exterior derivative respectively, $s(x, y)$ is the fundamental solution of Laplace equation

$$s(x,y) = \begin{cases} \dfrac{1}{2\pi}\log|x-y|, & \text{if } n = 2, \\ \dfrac{1}{(2-n)\omega_n}|x-y|^{2-n}, & \text{if } n \geq 3 \end{cases}$$

and $s_h(x, y)$ is the double h-form introduced by Hodge in [22]

$$s_h(x,y) = \sum_{j_1<\ldots<j_h} s(x,y)dx^{j_1}\ldots dx^{j_h}dy^{j_1}\ldots dy^{j_h}.$$

Since, for a scalar function f and for a fixed h, we have $*df \wedge dx^h = (-1)^{n-1}\partial_h f\, dx$, denoting by w the harmonic double layer potential with density $\phi \in W^{1,p}(\Sigma)$, (8) implies

$$\partial_h w(x) = -\Theta_h(d\varphi)(x), \quad x \in \Omega \tag{9}$$

where, for every $\psi \in L^p_1(\Sigma)$,

$$\Theta_h(\psi)(x) = *\left(\int_\Sigma d_x[s_{n-2}(x,\ y)]\wedge\psi(y)\wedge dx^h\right), \quad x \in \Omega. \tag{10}$$

The following lemma can be considered as an extension of formula (9) to elasticity. Here du denotes the vector $(du_1, ..., du_n)$ and $\psi = (\psi_1, ..., \psi_n)$ is an element of $[L_1^p(\Sigma)]^n$.

Lemma 3.1. *Let* w^ξ *be the double layer potential* (7) *with density* $u \in [W^{1,p}(\Sigma)]^n$. *Then*

$$\partial_s w_j^\xi(x) = \mathcal{K}_{js}^\xi(du)(x), \quad x \in \Omega,\ j,\ s = 1, \ldots, n, \tag{11}$$

where

$$\mathcal{K}_{js}^\xi(\psi)(x) = \Theta_s(\psi_j)(x) - \frac{1}{(n-2)!}\delta_{hij_3\ldots j_n}^{123\ldots n}\int_\Sigma \partial_{x_s}K_{hj}^\xi(x,\ y) \wedge \psi_i(y) \wedge dy^{j_3}\ldots dy^{j_n}, \tag{12}$$

$$K_{hj}^\xi(x,\ y) = \frac{1}{\omega_n}\frac{k(\xi+1)}{2(k+1)}\frac{(y_l-x_l)(y_j-x_j)}{|y-x|^n} + \frac{k-(2+k)\xi}{2(k+1)}\delta_{lj}s(x,\ y), \tag{13}$$

and Θ_h *is given by* (10), $h = 1, ..., n$.

Proof. Let $n \geq 3$. Denote by M^{hi} the tangential operators $M^{hi} = \nu_h\partial_i - \nu_i\partial_h$, h, $i = 1, ..., n$. By observing that

$$M^{hi}\left(\frac{x_hx_j}{|x|^n}\right) = \frac{\delta_{ij}x_h\nu_h}{|x|^n} - n\frac{x_ix_jx_h\nu_h}{|x|^{n+2}},$$

we find in Ω

$$\begin{aligned} w_j^\xi(x) = &-\frac{1}{\omega_n}\int_\Sigma u_i(y)\left\{\delta_{ij}\frac{(y_h-x_h)\nu_h(y)}{|y-x|^n} - \frac{k(\xi+1)}{2(k+1)}M_y^{hi}\left[\frac{(y_h-x_h)(y_j-x_j)}{|y-x|^n}\right]\right.\\ &\left.+\frac{k-(2+k)\xi}{2(k+1)}M_y^{ij}\left[\frac{|y-x|^{2-n}}{2-n}\right]\right\}d\sigma_y = \\ &-\int_\Sigma u_j(y)\frac{\partial s(x,y)}{\partial\nu_y}d\sigma_y + \int_\Sigma u_i(y)\left\{\frac{k(\xi+1)}{2(k+1)}M_y^{hi}\left[\frac{(y_h-x_h)(y_j-x_j)}{|y-x|^2}(2-n)s(x,\ y)\right]\right.\\ &\left.-\frac{k-(2+k)\xi}{2(k+1)}M_y^{ij}\left[s(x,\ y)\right]\right\}d\sigma_y. \end{aligned}$$

An integration by parts on Σ leads to

$$\begin{aligned} w_j^\xi(x) = &-\int_\Sigma u_j(y)\frac{\partial s(x,y)}{\partial\nu_y}d\sigma_y - \int_\Sigma M^{hi}[u_i(y)]\left\{\frac{k(\xi+1)(2-n)}{2(k+1)}\frac{(y_h-x_h)(y_j-x_j)}{|y-x|^2}\right.\\ &\left.+\frac{k-(2+k)\xi}{2(k+1)}\delta_{hj}\right\}s(x,\ y)d\sigma_y = -\int_\Sigma u_j(y)\frac{\partial s(x,y)}{\partial\nu_y}d\sigma_y - \int_\Sigma M^{hi}[u_i(y)]K_{hj}^\xi(x,\ y)d\sigma_y. \end{aligned}$$

Therefore, by recalling (9),

$$\partial_s w_j^\xi(x) = \Theta_s(du_j)(x) - \int_\Sigma M^{hi}[u_i(y)]\ \partial_{x_s}[K_{hj}^\xi(x,\ y)]d\sigma_y. \tag{14}$$

If f is a scalar function, we may write

$$M^{hi}(f)d\sigma = \frac{1}{(n-2)!}\delta_{hij_3\ldots j_n}^{123\ldots n}df \wedge dx^{j_3}\ldots\ dx^{j_n}.$$

This identity is established by observing that on Σ we have

$$\frac{1}{(n-2)!}\delta^{123\ldots n}_{hij_3\ldots j_n} df \wedge dx^{j_3}\ldots dx^{j_n} = \frac{1}{(n-2)!}\delta^{123\ldots n}_{hij_3\ldots j_n}\partial_{j_2} f dx^{j_2}\wedge\ldots dx^{j_n} =$$
$$\frac{1}{(n-2)!}\delta^{123\ldots n}_{hij_3\ldots j_n}\delta^{1\ldots n}_{j_1\ldots j_n}\nu_{j_1}\partial_{j_2} f\, d\sigma = \delta^{hi}_{j_1 j_2}\nu_{j_1}\partial_{j_2} f\, d\sigma = (\nu_h\partial_i f - \nu_i\partial_h f)d\sigma.$$

Then we can rewrite (14) as

$$\partial_s w^{\xi}_j(x) = \Theta_s(du_j)(x) - \frac{1}{(n-2)!}\delta^{123\ldots n}_{hij_3\ldots j_n}\int_\Sigma \partial_{x_s}[K^{\xi}_{hj}(x,\ y)]\wedge du_i(y)\wedge dy^{j_3}\ldots dy^{j_n}.$$

Similar arguments prove the result if $n = 2$. We omit the details. □

3.2 Some jump formulas

Lemma 3.2. *Let $f \in L^1(\Sigma)$. If $\eta \in \Sigma$ is a Lebesgue point for f, we have*

$$\begin{aligned}&\lim_{x\to\eta}\int_\Sigma f(y)\partial_{x_s}\frac{(y_p - x_p)(y_j - x_j)}{|y-x|^n}d\sigma_y = \\ &\frac{\omega_n}{2}(\delta_{pj} - 2\nu_j(\eta)\nu_p(\eta))\nu_s(\eta)f(\eta) + \int_\Sigma f(y)\partial_{x_s}\frac{(y_p-\eta_p)(y_j-\eta_j)}{|y-\eta|^n}d\sigma_y,\end{aligned}\tag{15}$$

where the limit has to be understood as an internal angular boundary value[1].

Proof. Let $h_{pj}(x) = x_p x_j |x|^{-n}$. Since $h \in C^\infty(\mathbb{R}^n\setminus\{0\})$ is even and homogeneous of degree $2 - n$, due to the results proved in [23], we have

$$\lim_{x\to\eta}\int_\Sigma f(y)\partial_{x_s}\frac{(y_p-x_p)(y_j-x_j)}{|y-x|^n}d\sigma_y = -\nu_s(\eta)\gamma_{pj}(\eta)f(\eta)+\int_\Sigma f(y)\partial_{x_s}\frac{(y_p-\eta_p)(y_j-\eta_j)}{|y-\eta|^n}d\sigma_y,\tag{16}$$

where $\gamma_{pj}(\eta) = -2\pi^2\mathcal{F}(h_{pj})(\nu_\eta)$, $\mathcal{F}$ being the Fourier transform

$$\mathcal{F}(h)(x) = \int_{\mathbb{R}^n} h(y)e^{-2\pi\, i x\cdot y}dy$$

(see also [24] and note that in [23,24]ν is the inner normal). On the other hand

$$\mathcal{F}(h_{pj})(x) = \frac{1}{2-n}\mathcal{F}(x_p\partial_j(|x|^{2-n})) = -\frac{1}{(2-n)2\pi i}\partial_p\mathcal{F}(\partial_j(|x|^{2-n})) = -\frac{1}{2-n}\partial_p(x_j\mathcal{F}(|x|^{2-n}))$$

and, since

$$\mathcal{F}(|x|^{2-n}) = \frac{\pi^{n/2-2}}{\Gamma(n/2-1)}|x|^{-2}$$

(see, e.g., [[25], p. 156]), we find

$$\mathcal{F}(h_{pj})(x) = \frac{\pi^{n/2-2}}{(n-2)\Gamma(n/2-1)}\partial_p(x_j|x|^{-2}) = \frac{\pi^{n/2-2}}{(n-2)\Gamma(n/2-1)}(\delta_{pj}|x|^{-2} - 2x_j x_p|x|^{-4}).$$

Finally, keeping in mind that $\omega_n = n\,\pi^{n/2}/\Gamma(n/2 + 1)$ and $\Gamma(n/2 + 1) = n(n - 2)\Gamma(n/2 - 1)/4$, we obtain

$$\gamma_{pj}(\eta) = -2\frac{\pi^{n/2}}{(n-2)\Gamma(n/2-1)}(\delta_{pj} - 2\nu_j(\eta)\nu_p(\eta)) = -\frac{\omega_n}{2}(\delta_{pj} - 2\nu_j(\eta)\nu_p(\eta)).$$

Combining this formula with (16) we get (15). □

Lemma 3.3. *Let $\psi \in L^p_1(\Sigma)$. Let us write ψ as $\psi = \psi_h dx^h$ with*

$$\nu_h \psi_h = 0. \tag{17}$$

Then, for almost every $\eta \in \Sigma$,

$$\lim_{x\to\eta} \Theta_s(\psi)(x) = -\frac{1}{2}\psi_s(\eta) + \Theta_s(\psi)(\eta), \tag{18}$$

where Θ_s is given by (10) *and the limit has to be understood as an internal angular boundary value.*

Proof. First we note that the assumption (17) is not restrictive, because, given the 1-form ψ on Σ, there exist scalar functions ψ_h defined on Σ such that $\psi = \psi_h dx^h$ and (17) holds (see [[26], p. 41]). We have

$$\Theta_s(\psi)(x) = \sum_{j_1<\ldots<j_{n-2}} *\left(\int_\Sigma \partial_{x_i}[s(x,\ y)]\psi_h(y)dy^{j_1}\ldots dy^{j_{n-2}}dy^h dx^i dx^{j_1}\ldots dx^{j_{n-2}}dx^s\right) =$$
$$\sum_{j_1<\ldots<j_{n-2}} \delta^{12\ldots\ldots\ldots n}_{kj_1\ldots j_{n-2}h}\delta^{12\ldots\ldots\ldots n}_{ij_1\ldots j_{n-2}s}\int_\Sigma \partial_{x_i}[s(x,\ y)]\nu_k(y)\psi_h(y)d\sigma_y = \delta^{is}_{kh}\int_\Sigma \partial_{x_i}[s(x,\ y)]\nu_k(y)\psi_h(y)d\sigma_y$$

and then

$$\lim_{x\to\eta} \Theta_s(\psi)(x) = -\frac{1}{2}\delta^{is}_{kh}\nu_i(\eta)\nu_k(\eta)\psi_h(\eta) + \Theta_s(\psi)(\eta)$$

a.e. on Σ. From (17) it follows that $\delta^{is}_{kh}\nu_i\nu_k\psi_h = \nu_i\nu_i\psi_s - \nu_i\nu_s\psi_i = \psi_s$ and (18) is proved. □

Lemma 3.4. *Let $\psi \in L^p_1(\Sigma)$. Let us write ψ as $\psi = \psi_h dx^h$ and suppose that* (17) *holds. Then, for almost every $\eta \in \Sigma$,*

$$\begin{gathered}\lim_{x\to\eta}\frac{1}{(n-2)!}\delta^{123\ldots n}_{lij_3\ldots j_n}\int_\Sigma \partial_{x_s}K^\xi_{lj}(x,\ y)\wedge\psi(y)\wedge dy^{j_3}\ldots dy^{j_n} = \\ -\left[\frac{k-\xi}{2(k+1)}\nu_j(\eta)\psi_i(\eta) + \frac{\xi}{2}\nu_i(\eta)\psi_j(\eta)\right]\nu_s(\eta) + \frac{1}{(n-2)!}\delta^{123\ldots n}_{lij_3\ldots j_n}\int_\Sigma \partial_{x_s}K^\xi_{lj}(\eta,\ y)\wedge\psi(y)\wedge dy^{j_3}\ldots dy^{j_n},\end{gathered} \tag{19}$$

where K^ξ is defined by (13) *and the limit has to be understood as an internal angular boundary value.*

Proof. We have

$$\frac{1}{(n-2)!}\delta^{123\ldots n}_{lij_3\ldots j_n}\int_\Sigma \partial_{x_s}K^\xi_{lj}(x,\ y)\wedge\psi(y)\wedge dy^{j_3}\ldots dy^{j_n} =$$
$$\frac{1}{(n-2)!}\delta^{123\ldots n}_{lij_3\ldots j_n}\delta^{123\ldots n}_{rhj_3\ldots j_n}\int_\Sigma \partial_{x_s}K^\xi_{lj}(x,\ y)\psi_h(y)\nu_r(y)d\sigma_y =$$
$$\delta^{li}_{rh}\int_\Sigma \partial_{x_s}K^\xi_{lj}(x,\ y)\psi_h(y)\nu_r(y)d\sigma_y.$$

Keeping in mind (13), formula (15) leads to

$$\lim_{x\to\eta}\frac{1}{(n-2)!}\delta^{123\ldots n}_{lij_3\ldots j_n}\int_\Sigma \partial_{x_s}K^\xi_{lj}(x,\ y)\wedge\psi(y)\wedge dy^{j_3}\ldots dy^{j_n} =$$
$$\delta^{li}_{rh}\left[\frac{k(\xi+1)}{4(k+1)}(\delta_{lj} - 2\nu_j(\eta)\nu_l(\eta))\nu_s(\eta) - \frac{k-(2+k)\xi}{4(k+1)}\delta_{lj}\nu_s(\eta)\right]\nu_r(\eta)\psi_h(\eta)$$
$$+\frac{1}{(n-2)!}\delta^{123\ldots n}_{lij_3\ldots j_n}\int_\Sigma \partial_{x_s}K^\xi_{lj}(\eta,\ y)\wedge\psi(y)\wedge dy^{j_3}\ldots dy^{j_n}.$$

On the other hand

$$\delta^{li}_{rh}\left[\frac{k(\xi+1)}{4(k+1)}(\delta_{lj}-2\nu_j\nu_l)\nu_s-\frac{k-(2+k)\xi}{4(k+1)}\delta_{lj}\nu_s\right]\nu_r\psi_h=\delta^{li}_{rh}\left[\frac{\xi}{2}\delta_{lj}\nu_s-\frac{k(\xi+1)}{2(k+1)}\nu_j\nu_l\nu_s\right]\nu_r\psi_h=$$
$$\left[\frac{\xi}{2}\delta_{lj}\nu_s-\frac{k(\xi+1)}{2(k+1)}\nu_j\nu_l\nu_s\right](\nu_l\psi_i-\nu_i\psi_l)=-\frac{k-\xi}{2(k+1)}\nu_j\nu_s\psi_i-\frac{\xi}{2}\nu_i\nu_s\psi_j,$$

and the result follows. □

Lemma 3.5. *Let $\psi=(\psi_1,\ \dots,\ \psi_n)\in[L^p_1(\Sigma)]^n$. Then, for almost every $\eta\in\Sigma$,*

$$\lim_{x\to\eta}[(k-\xi)\mathcal{K}^{\xi}_{jj}(\psi)(x)\nu_i(\eta)+\nu_j(\eta)\mathcal{K}^{\xi}_{ij}(\psi)(x)+\xi\nu_j(\eta)\mathcal{K}^{\xi}_{ji}(\psi)(x)]=$$
$$(k-\xi)\mathcal{K}^{\xi}_{jj}(\psi)(\eta)\nu_i(\eta)+\nu_j(\eta)\mathcal{K}^{\xi}_{ij}(\psi)(\eta)+\xi\nu_j(\eta)\mathcal{K}^{\xi}_{ji}(\psi)(\eta), \tag{20}$$

$\mathcal{K}^{\xi}$being as in (12) *and the limit has to be understood as an internal angular boundary value.*

Proof. Let us write ψ_i as $\psi_i=\psi_{ih}dx^h$ with

$$\nu_h\psi_{ih}=0,\quad i=1,\ \dots,\ n. \tag{21}$$

In view of Lemmas 3.3 and 3.4 we have

$$\lim_{x\to\eta}\mathcal{K}^{\xi}_{js}(\psi)(x)=-\frac{1}{2}\psi_{js}(\eta)+\left[\frac{k-\xi}{2(k+1)}\nu_j(\eta)\psi_{hh}(\eta)+\frac{\xi}{2}\nu_h(\eta)\psi_{hj}(\eta)\right]\nu_s(\eta)+\mathcal{K}^{\xi}_{js}(\psi)(\eta).$$

Therefore

$$\lim_{x\to\eta}[(k-\xi)\mathcal{K}^{\xi}_{jj}(\psi)(x)\nu_i(\eta)+\nu_j(\eta)\mathcal{K}^{\xi}_{ij}(\psi)(x)+\xi\nu_j(\eta)\mathcal{K}^{\xi}_{ji}(\psi)(x)]=$$
$$\Phi(\psi)(\eta)+(k-\xi)\mathcal{K}^{\xi}_{jj}(\psi)(\eta)\nu_i(\eta)+\nu_j(\eta)\mathcal{K}^{\xi}_{ij}(\psi)(\eta)+\xi\nu_j(\eta)\mathcal{K}^{\xi}_{ji}(\psi)(\eta),$$

where

$$\Phi(\psi)=(k-\xi)\left[-\frac{1}{2}\psi_{jj}+\left(\frac{k-\xi}{2(k+1)}\nu_j\psi_{hh}+\frac{\xi}{2}\nu_h\psi_{hj}\right)\nu_j\right]\nu_i$$
$$+\nu_j\left[-\frac{1}{2}\psi_{ij}+\left(\frac{k-\xi}{2(k+1)}\nu_i\psi_{hh}+\frac{\xi}{2}\nu_h\psi_{hi}\right)\nu_j\right]+\xi\nu_j\left[-\frac{1}{2}\psi_{ji}+\left(\frac{k-\xi}{2(k+1)}\nu_j\psi_{hh}+\frac{\xi}{2}\nu_h\psi_{hj}\right)\nu_i\right].$$

Conditions (21) lead to

$$\Phi(\psi)=-\frac{1}{2}\left[(k-\xi)\left(1-\frac{k-\xi}{k+1}\right)-\frac{k-\xi}{k+1}-\xi\frac{k-\xi}{k+1}\right]\nu_i\psi_{hh}.$$

The bracketed expression vanishing, $\Phi=0$ and the result is proved. □

Remark 3.6. In Lemmas 3.2, 3.3, 3.4 and 3.5 we have considered internal angular boundary values. It is clear that similar formulas hold for external angular boundary values. We have just to change the sign in the first term on the right hand sides in (15), (18) and (19), while (20) remains unchanged.

3.3 Reduction of a certain singular integral operator

The results of the previous subsection imply the following lemmas.

Lemma 3.7. *Let w^{ξ} be the double layer potential* (7) *with density $u\in[W^{1,p}(\Sigma)]^n$. Then*

$$L^{\xi}_{+,i}(w^{\xi})=L^{\xi}_{-,i}(w^{\xi})=(k-\xi)\mathcal{K}^{\xi}_{jj}(du)\nu_i+\nu_j\mathcal{K}^{\xi}_{ij}(du)+\xi\nu_j\mathcal{K}^{\xi}_{ji}(du) \tag{22}$$

a.e. on Σ, *where* $L_+^\xi(w^\xi)$*and* $L_-^\xi(w^\xi)$*denote the internal and the external angular boundary limit of* $L^\xi(w^\xi)$ *respectively and* $\mathcal{K}^\xi$*is given by* (12).

Proof. It is an immediate consequence of (11), (20) and Remark 3.6. □

Remark 3.8. The previous result is connected to [[1], Theorem 8.4, p. 320].

Lemma 3.9. *Let* $R : [L^p(\Sigma)]^n \to [L_1^p(\Sigma)]^n$*be the following singular integral operator*

$$R\varphi(x) = \int_\Sigma d_x[\Gamma(x,\ y)]\varphi(y)d\sigma_y. \tag{23}$$

Let us define $R'^\xi : [L_1^p(\Sigma)]^n \to [L^p(\Sigma)]^n$*to be the singular integral operator*

$$R'^\xi_i(\psi)(x) = (k-\xi)\mathcal{K}^\xi_{jj}(\psi)(x)\nu_i(x) + \nu_j(x)\mathcal{K}^\xi_{ij}(\psi)(x) + \xi\nu_j(x)\mathcal{K}^\xi_{ji}(\psi)(x). \tag{24}$$

Then

$$R'^\xi R\varphi = -\frac{1}{4}\varphi + (T^\xi)^2\varphi, \tag{25}$$

where

$$T^\xi\varphi(x) = \int_\Sigma L^\xi_x[\Gamma(x,y)]\varphi(y)d\sigma_y. \tag{26}$$

Proof. Let u be the simple layer potential with density $\phi \in [L^p(\Sigma)]^n$. In view of Lemma 3.7, we have a.e. on Σ

$$R'^\xi_i(R\varphi) = (k-\xi)\mathcal{K}^\xi_{jj}(du)\nu_i + \nu_j\mathcal{K}^\xi_{ij}(du) + \xi\nu_j\mathcal{K}^\xi_{ji}(du) = L^\xi_i(w^\xi),$$

where w^ξ is the double layer potential (7) with density u. Moreover, if $x \in \Omega$,

$$w^\xi_j(x) = \int_\Sigma u_i(y)\ L^\xi_{i,y}[\Gamma^j(x,y)]d\sigma_y = -u_j(x) + \int_\Sigma L^\xi_i[u(y)]\ \Gamma_{ij}(x,y)\ d\sigma_y$$

and then, on account of (26),

$$L^\xi w^\xi = -\frac{1}{2}L^\xi u + T^\xi(L^\xi u) = -\frac{1}{2}\left(\frac{1}{2}\varphi + T^\xi\varphi\right) + T^\xi\left(\frac{1}{2}\varphi + T^\xi\varphi\right) = -\frac{1}{4}\varphi + (T^\xi)^2\varphi.$$

□

Corollary 3.10. *The operator R defined by* (23) *can be reduced on the left. A reducing operator is given by* R'^ξ *with* $\xi = k/(2+k)$.

Proof. This follows immediately from (25), because of the weak singularity of the kernel in (26) when $\xi = k/(2+k)$ (see (6)). □

3.4 The dimension of some eigenspaces

Let T be the operator defined by (26) with $\xi = 1$, i.e.

$$T\varphi(x) = \int_\Sigma L_x[\Gamma(x,y)]\varphi(y)d\sigma_y, \quad x \in \Sigma, \tag{27}$$

and denote by T^* its adjoint.

In this subsection we determine the dimension of the following eigenspaces

$$\mathcal{V}_\pm = \left\{\varphi \in [L^p(\Sigma)]^n : \mp\frac{1}{2}\varphi + T^*\varphi = 0\right\}; \tag{28}$$

$$\mathcal{W}_{\pm} = \left\{ \varphi \in [L^p(\Sigma)]^n : \pm\frac{1}{2}\varphi + T\varphi = 0 \right\}. \tag{29}$$

We first observe that the (total) indices of singular integral systems in (28)-(29) vanish. This can be proved as in [[1], pp. 235-238]. Moreover, by standard techniques, one can prove that all the eigenfunctions are hölder-continuous and then these eigenspaces do not depend on p. This implies that

$$\dim \mathcal{V}_{+} = \dim \mathcal{W}_{-}, \quad \dim \mathcal{V}_{-} = \dim \mathcal{W}_{+}. \tag{30}$$

The next two lemmas determine such dimensions. Similar results for Laplace equation can be found in [[27], Chapter 3].

Lemma 3.11. *The spaces* $\mathcal{V}_{+}$*and* $\mathcal{W}_{-}$*have dimension* $n(n+1)m/2$. *Moreover*

$$\mathcal{V}_{+} = \{v_h \mathcal{X}\Sigma_j : h = 1, \dots, n(n+1)/2, j = 1, \dots, m\},$$

where $\{v_h : h = 1 ..., n(n+1)/2\}$ *is an orthonormal basis of the space* $\mathcal{R}$*and* $\mathcal{X}\Sigma_j$*is the characteristic function of* Σ_j.

Proof. We define the vector-valued functions α_j, $j = 1, ..., m$ as $\alpha_j(x) = (a + Bx)\mathcal{X}_{\Sigma_j}(x)$, $x \in \Sigma$. For a fixed $j = 1, ..., m$, the function $\alpha_j(x)$ belongs to $\mathcal{V}_{+}$; indeed

$$-\frac{1}{2}(a+Bx)\mathcal{X}_{\Sigma_j}(x) + \int_{\Sigma} [L_y\Gamma(x,y)]'(a+By)\mathcal{X}_{\Sigma_j}(y)d\sigma_y = -\frac{1}{2}(a+Bx)\mathcal{X}_{\Sigma_j}(x) + \int_{\Sigma_j} [L_y\Gamma(x,y)]'(a+By)d\sigma_y =$$
$$-\frac{1}{2}(a+Bx) + \frac{1}{2}(a+Bx) = 0, \quad x \in \Sigma_j,$$

because of

$$\int_{\Sigma} [L_y\Gamma(x,y)]'\alpha_j(y)d\sigma_y = \begin{cases} \alpha_j(x) & x \in \Omega_j, \\ \alpha_j(x)/2 & x \in \Sigma_j, \\ 0 & x \notin \overline{\Omega}_j. \end{cases} \tag{31}$$

Now we prove that the following $n(n+1)m/2$ eigensolutions of $\mathcal{V}_{+}$

$$w_{hj}(x) = v_h(x)\mathcal{X}_{\Sigma_j}(x), \quad h = 1, \dots, n(n+1)/2, j = 1, \dots, m, x \in \Sigma$$

are linearly independent. Indeed, if $\sum_{h=1}^{n(n+1)/2} \sum_{j=1}^{m} c_{hj}w_{hj} = 0$, we have

$$\sum_{h=1}^{n(n+1)/2} c_{hj}v_h(x) = 0, \quad x \in \Sigma_j, j = 1, \dots, m.$$

Then, by applying a classical uniqueness theorem to the domain Ω_j,

$$\sum_{h=1}^{n(n+1)/2} c_{hj}v_h(x) = 0, \quad x \in \Omega_j, j = 1, \dots, m,$$

from which it easily follows that

$$c_{hj} = 0, \quad h = 1, \dots, n(n+1)/2, j = 1, \dots, m.$$

Thus, $\dim \mathcal{V}_{+} \geq n(n+1)m/2$. On the other hand, suppose $\varphi \in \mathcal{W}_{-}$ and let u be the simple layer potential with density ϕ. Since $E_u = 0$ in Ω_j and $L_{-}u = 0$ on Σ_j, $u = a^j + B^j x$ on each connected component Ω_j, $j = 1, ..., m$, and $u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}_0$. Note that this is true also for $n = 2$, because $\varphi \in \mathcal{W}_-$ implies $\int_\Sigma \varphi\, d\sigma = 0$. We can define a linear map τ as follows

$$\tau : \mathcal{W}_- \to (\mathbb{R}^n \times \mathscr{S}_n)^m$$
$$\varphi \to (a^1, B^1, \dots, a^m, B^m).$$

If $\tau(\phi) = 0$, from a classical uniqueness theorem, we have that $\phi \equiv 0$ in $\mathbb{R}^n$. Thus, τ is an injective map and $\dim \mathcal{W}_- \le n(n+1)m/2$. The assertion follows from (30). □

Lemma 3.12. *The spaces $\mathcal{V}_-$and $\mathcal{W}_+$have dimension $n(n + 1)/2$. Moreover $\mathcal{V}_-$is constituted by the restrictions to Σ of the rigid displacements.*

Proof. Let $\alpha \in \mathcal{R}$. If $x \in \Sigma$, we have

$$\frac{1}{2}\alpha(x) + \int_\Sigma [L_y\Gamma(x,y)]'\alpha(y)d\sigma_y = \frac{1}{2}\alpha(x) - \frac{1}{2}\alpha(x) = 0,$$

thanks to

$$\int_\Sigma [L_y\Gamma(x,y)]'\alpha(y)d\sigma_y = \begin{cases} -\alpha(x) & x \in \Omega, \\ -\alpha(x)/2 & x \in \Sigma, \\ 0 & x \notin \overline{\Omega}. \end{cases}$$

This shows that the restriction to Σ of α belongs to $\mathcal{V}_-$ and then $\dim \mathcal{V}_- \ge \dim \mathcal{R} = n(n+1)/2$. On the other hand, suppose $\phi \in \mathcal{W}_+$ and let u be the simple layer potential with density φ. Since $Eu = 0$ in Ω and $L_+u = 0$ on Σ, $u = a + Bx$ in Ω. Let σ be the linear map

$$\sigma : \mathcal{W}_+ \to \mathbb{R}^n \times\ l_n$$
$$\phi \to (a, B).$$

If $n \ge 3$, we have that $\sigma(\varphi) = 0$ implies $u \equiv 0$ in $\mathbb{R}^n$ and then $\varphi \equiv 0$ on Σ, in view of classical uniqueness theorems.

If $n = 2$, define $\mathcal{W}_+^0 = \left\{\phi \in \mathcal{W}_+ / \int_\Sigma \phi\, d\sigma = 0\right\}$. We have $\sigma|_{\mathcal{W}_+^0}$ is injective and its range does not contain the vectors $((1, 0), 0)$ and $((0, 1), 0)^2$. Therefore $\dim \mathcal{W}_+^0 \le 1$. On the other hand, $\dim \mathcal{W}_+ - 2 \le \dim \mathcal{W}_+^0$and then $\dim \mathcal{W}_+ \le 3$. In any case, $\dim \mathcal{W}_+ \le n(n+1)/2$ and the result follows from (30). □

4 The bidimensional case

The case $n = 2$ requires some additional considerations. It is well-known that there are some domains in which no every harmonic function can be represented by means of a harmonic simple layer potential. For instance, on the unit disk we have

$$\int_{|y|=1} \log|x-y|ds_y = 0, \quad |x| < 1.$$

Similar domains occur also in elasticity. In order to give explicitly such an example, let us prove the following lemma.

Lemma 4.1. *Let Σ_R be the circle of radius R centered at the origin. We have*

$$\int_{\Sigma_R} |x-y|^2 \log|x-y|ds_y = 2\pi R(R^2\log R + (1+\log R)|x|^2), \quad |x| < R. \tag{32}$$

Proof. Denote by $u(x)$ the function on the left hand side of (32) and by Ω_R the ball of radius R centered at the origin. Let us fix $x_0 \in \Sigma_R$. For any $x \in \Sigma_R$ we have

$$\int_{\Sigma_R} |x-y|^2 \log|x-y| ds_y = \int_{\Sigma_R} |x_0-y|^2 \log|x_0-y| ds_y$$

and then u is constant on Σ_R. Moreover

$$\Delta u(x) = 4\int_{\Sigma_R} (1+\log|x-y|) ds_y$$

and then also Δu is constant on Σ_R. Since Δu is harmonic in Ω_R and continuous on $\overline{\Omega}_R$, it is constant in Ω_R and then

$$\Delta u(x) = \Delta u(0) = 4\int_{\Sigma_R} (1+\log|y|) ds_y = 8\pi R(1+\log R), \quad x \in \Omega_R.$$

The function $u(x)$ - $2\pi R(1 + \log R)\ |x|^2$ is continuous on $\overline{\Omega}_R$, harmonic in Ω_R and constant on Σ_R. Then it is constant in Ω_R and

$$u(x) - 2\pi R(1+\log R)|x|^2 = u(0) = \int_{\Sigma_R} |y|^2 \log|y| d\sigma_y = 2\pi R^3 \log R.$$

□

Corollary 4.2. *Let Σ_R be the circle of radius R centered at the origin. We have*

$$\int_{\Sigma_R} \Gamma_{ij}(x,y) ds_y = \delta_{ij} \frac{R}{4(k+1)}(k - 2(k+2)\log R), \quad |x| < R. \tag{33}$$

Proof. Since

$$\partial_{11}\int_{\Sigma_R} |x-y|^2 \log|x-y| ds_y = 2\int_{\Sigma_R} \log|x-y| ds_y + 2\int_{\Sigma_R} \frac{(x_1-y_1)^2}{|x-y|^2} ds_y + 2\pi R,$$

formula (32) implies

$$\int_{\Sigma_R} \frac{(x_1-y_1)^2}{|x-y|^2} ds_y = \pi R, \quad |x| < R.$$

In a similar way

$$\int_{\Sigma_R} \frac{(x_2-y_2)^2}{|x-y|^2} ds_y = \pi R, \quad |x| < R.$$

From (32) we have also

$$\partial_{12}\int_{\Sigma_R} |x-y|^2 \log|x-y| ds_y = 2\int_{\Sigma_R} \frac{(x_1-y_1)(x_2-y_2)}{|x-y|^2} ds_y = 0, \quad |x| < R.$$

Keeping in mind the expression (1), (33) follows. □

This corollary shows that, if $R = \exp[k/(2(k + 2))]$, we have

$$\int_{\Sigma_R} \Gamma(x,y) e_1 ds_y = \int_{\Sigma_R} \Gamma(x,y) e_2 ds_y = 0, \quad |x| < R.$$

This implies that in Ω_R, for such a value of R, we cannot represent any smooth solution of the system $E_u = 0$ by means of a simple layer potential.

If there exists some constant vector which cannot be represented in the simply connected domain Ω by a simple layer potential, we say that the boundary of Ω is *exceptional*. We have proved that

Lemma 4.3. *The circle* Σ_R *with* $R = \exp[k/(2(k+2))]$ *is exceptional for the operator* $\Delta + k\nabla div$.

Due to the results in [28], one can scale the domain in such a way that its boundary is not exceptional.

Here we show that also in some $(m + 1)$-connected domains one cannot represent any constant vectors by a simple layer potential and that this happens if, and only if, the exterior boundary Σ_0 (considered as the boundary of the simply connected domain Ω_0) is exceptional.

We note that, if any constant vector c can be represented by a simple layer potential, then any sufficiently smooth solution of the system $Eu = 0$ can be represented by a simple layer potential as well (see Section 5 below).

We first prove a property of the singular integral system

$$\int_\Sigma \varphi_j(y)\frac{\partial}{\partial s_x}\Gamma_{ij}(x,y)ds_y = 0, \quad x \in \Sigma,\ i = 1, 2. \tag{34}$$

Lemma 4.4. *Let* $\Omega \subset \mathbb{R}^2$ *be an* $(m + 1)$*-connected domain. Denote by* $\mathcal{P}$*the eigenspace in* $[L^p(\Sigma)]^2$ *of the system* (34). *Then* $\dim\mathcal{P} = 2(m+1)$.

Proof. We have

$$\frac{\partial\Gamma_{ij}}{\partial s_x}(x,y) = \frac{1}{2\pi}\frac{\partial}{\partial s_x}\left(-\frac{(k+2)\delta_{ij}}{2(k+1)}\log|x-y| + \frac{k}{2(k+1)}\frac{(y_i-x_i)(y_j-x_j)}{|x-y|^2}\right)ds_y$$

and, since

$$\begin{aligned}\frac{\partial}{\partial s_x}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2} &= \dot{x}_i\frac{x_j-y_j}{|x-y|^2} + \dot{x}_j\frac{x_i-y_i}{|x-y|^2} - 2\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\frac{\partial}{\partial s_x}|x-y| = \\ &\dot{x}_i\frac{\partial}{\partial x_j}\log|x-y| + \dot{x}_j\frac{\partial}{\partial x_i}\log|x-y| - 2\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\frac{\partial}{\partial s_x}\log|x-y| = \\ &2\left(\dot{x}_i\dot{x}_j - \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right)\frac{\partial}{\partial s_x}\log|x-y| + \mathcal{O}(|y-x|^{h-1})\end{aligned}$$

(the dot denotes the derivative with respect to the arc length on Σ), we find[3]

$$\frac{\partial}{\partial s_x}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2} = \mathcal{O}(|y-x|^{h-1}).$$

We have proved that[4]

$$\frac{\partial}{\partial s_x}\Gamma_{ij}(x,y) = -\frac{1}{2\pi}\frac{k+2}{2(k+1)}\delta_{ij}\frac{\partial}{\partial s_x}\log|x-y| + \mathcal{O}(|y-x|^{h-1})$$

and then the system (34) is of regular type (see [15,29]). From the general theory we know that such a system can be regularized to a Fredholm one. Let us consider now the adjoint system

$$\int_\Sigma \varphi_j(y)\frac{\partial}{\partial s_y}\Gamma_{ij}(x,y)ds_y = 0, \quad x \in \Sigma, i = 1, 2. \tag{35}$$

It is not difficult to see that the index is zero and then systems (34) and (35) have the same number of eigensolutions.

The vectors $e_i \mathcal{X}_{\Sigma_j}(i = 1, 2, j = 0, 1, \dots, m)$ are the only linearly independent eigensolutions of (35). Indeed it is obvious that such vectors satisfy the system (35). On the other hand, if ψ satisfies the system (35) then

$$\int_\Sigma \psi \frac{\partial f}{\partial s} ds = 0$$

for any $f \in [C^\infty(\mathbb{R}^2)]^2$. This can be siproved by the same method in [[13], pp. 189-190]. Therefore ψ has to be constant on each curve Σ_j ($j = 0, \dots, m$), i.e. ψ is a linear combination of $e_i \mathcal{X}_{\Sigma_j}(i = 1, 2, j = 0, 1, \dots, m)$. □

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^2$ be an $(m + 1)$-connected domain. The following conditions are equivalent:*

I. there exists a Hölder continuous vector function $\varphi \not\equiv 0$such that

$$\int_\Sigma \Gamma(x, y)\varphi(y) ds_y = 0, \quad x \in \Sigma; \tag{36}$$

II. there exists a constant vector which cannot be represented in Ω by a simple layer potential (i.e., there exists $c \in \mathbb{R}^2$ such that $c \notin \mathcal{S}^p$);

III. Σ_0 is exceptional;

IV. let $\phi_1, \dots, \phi_{2m+2}$ be linearly independent functions of $\mathcal{P}$and let $c_{jk} = (\alpha_{jk}, \beta_{jk}) \in \mathbb{R}^2$ be given by

$$\int_\Sigma \Gamma(x, y)\varphi_j(y) ds_y = c_{jk}, \quad x \in \Sigma_k, \ j = 1, \dots, 2m + 2, k = 0, \ 1, \dots, m.$$

Then

$$\det \mathcal{C} = 0, \tag{37}$$

where

$$C = \begin{pmatrix} \alpha_{1,0} & \cdots & \alpha_{2m+2,0} \\ \cdots & \cdots & \cdots \\ \alpha_{1,m} & \cdots & \alpha_{2m+2,m} \\ \beta_{1,0} & \cdots & \beta_{2m+2,0} \\ \cdots & \cdots & \cdots \\ \beta_{1,m} & \cdots & \beta_{2m+2,m} \end{pmatrix}.$$

Proof. I ⇒ II. Let u be the simple layer potential (3) with density ϕ.

Since $u = 0$ in Ω, and then on Σ_k, we find that $u = 0$ also in Ω_k ($k = 1, \dots, m$) in view of a known uniqueness theorem.

On the other hand $L_+u - L_-u = \phi$ on Σ and $\phi = 0$ on Σ_k, $k = 1, \dots, m$. This means that

$$\int_{\Sigma_0} \Gamma(x, y)\varphi(y) ds_y = 0, \quad x \in \Omega_0.$$

If II is not true, we can find two linear independent vector functions ψ_1 and ψ_2 such that

$$\int_\Sigma \Gamma(x, y)\psi_j(y) ds_y = e_j, \quad x \in \Omega, j = 1, 2.$$

Arguing as before, we find $\psi_j = 0$ on Σ_k, $k = 1, ..., m$, $j = 1, 2$, and then

$$\int_{\Sigma_0} \Gamma(x,y)\psi_j(y)ds_y = e_j, \quad x \in \Omega_0, j = 1, 2.$$

Since ϕ, ψ_1, ψ_2 belong to the kernel of the system

$$\int_{\Sigma_0} \frac{\partial}{\partial s_x}\Gamma(x,y)\psi(y)ds_y = 0, \quad x \in \Sigma_0,$$

Lemma 4.4 shows that they are linearly dependent. Let λ, μ_1, $\mu_2 \in \mathbb{R}$ such that $(\lambda, \mu_1, \mu_2) \neq (0, 0, 0)$ and

$$\lambda\varphi + \mu_1\psi_1 + \mu_2\psi_2 = 0 \quad \text{on } \Sigma_0. \tag{38}$$

This implies

$$\int_{\Sigma_0} \Gamma(x,y)(\lambda\varphi(y) + \mu_1\psi_1(y) + \mu_2\psi_2(y))ds_y = 0, \quad x \in \Omega_0,$$

i.e. $\mu_1 e_1 + \mu_2 e_2 = 0$, and then $\mu_1 = \mu_2 = 0$. Now (38) leads to $\lambda\phi = 0$ and thus $\lambda = 0$, which is absurd.

II $\Rightarrow$ III. If Σ_0 is not exceptional, for any $c \in \mathbb{R}^2$ there exists $\varrho \in [C^\lambda(\Sigma_0)]^2$ such that

$$\int_{\Sigma_0} \Gamma(x,y)\varrho(y)ds_y = c, \quad x \in \Omega_0.$$

Setting

$$\varphi(y) = \begin{cases} \varrho(y) & y \in \Sigma_0, \\ 0 & y \in \Sigma \setminus \Sigma_0, \end{cases}$$

we can write

$$\int_{\Sigma} \Gamma(x,y)\varphi(y)ds_y = c, \quad x \in \Omega,$$

and this contradicts II.

III $\Rightarrow$ IV. Let us suppose $\det \mathcal{C} \neq 0$. For any $c = (\alpha, \beta) \in \mathbb{R}^2$ there exists $\lambda = (\lambda_1, ..., \lambda_{2m+2})$ solution of the system

$$\sum_{j=1}^{2m+2} \lambda_j \alpha_{jk} = \alpha, \quad \sum_{j=1}^{2m+2} \lambda_j \beta_{jk} = \beta, \quad k = 0, \ldots, m,$$

i.e.

$$\sum_{j=1}^{2m+2} \lambda_j c_{jk} = c, \quad k = 0, \ldots, m.$$

Therefore

$$\int_{\Sigma} \Gamma(x,y) \sum_{j=1}^{2m+2} \lambda_j \varphi_j(y)ds_y = c, \quad x \in \Sigma.$$

Arguing as before, this leads to $\sum_{j=1}^{2m+2} \lambda_j \varphi_j = 0$ on Σ_k for $k = 1, ..., m$. Then Σ_0 is not exceptional.

IV ⇒ I. From (37) it follows that there exists an eigensolution $\lambda = (\lambda_1, ..., \lambda_{2m+2})$ of the homogeneous system

$$\sum_{j=1}^{2m+2} \lambda_j c_{jk} = 0, \quad k = 0, \ldots, m.$$

Set

$$\varphi(x) = \sum_{j=1}^{2m+2} \lambda_j \varphi_j(x).$$

In view of the linear independence of $\phi_1, ..., \phi_{2m+2}$, the vector function ϕ does not identically vanish and it is such that (36) holds. □

Definition 4.6. *Whenever $n = 2$ and Σ_0 is exceptional, we say that u belongs to $\mathcal{S}^p$if, and only if,*

$$u(x) = \int_\Sigma \Gamma(x, y)\varphi(y) ds_y + c, \quad x \in \Omega, \tag{39}$$

where $\phi \in [L^p(\Sigma)]^2$ and $c \in \mathbb{R}^2$.

5 The Dirichlet problem

The purpose of this section is to represent the solution of the Dirichlet problem in an $(m + 1)$-connected domain by means of a simple layer potential. Precisely we give an existence and uniqueness theorem for the problem

$$\begin{cases} u \in \mathcal{S}^p, \\ Eu = 0 \quad \text{in } \Omega, \\ u = f \quad \text{on } \Sigma, \end{cases} \tag{40}$$

where $f \in [W^{1,p}(\Sigma)]^n$.

We establish some preliminary results.

Theorem 5.1. *Given $\omega \in [L_1^p(\Sigma)]^n$, there exists a solution of the singular integral system*

$$\int_\Sigma d_x[\Gamma(x, y)]\varphi(y) d\sigma_y = \omega(x), \quad \varphi \in [L^p(\Sigma)]^n, \ x \in \Sigma \tag{41}$$

if, and only if,

$$\int_\Sigma \gamma \wedge \omega_i = 0, \quad i = 1, \ldots, n \tag{42}$$

for every $\gamma \in L_{n-2}^q(\Sigma)$ $(q = p/(p-1))$such that γ is a weakly closed $(n - 2)$-form.

Proof. Denote by $R^* : [L_{n-2}^q(\Sigma)]^n \to [L^q(\Sigma)]^n$ the adjoint of R (see (23)), i.e. the operator whose components are given by

$$R_j^* \psi(x) = \int_\Sigma \psi_i(y) \wedge d_y[\Gamma_{ij}(x, y)], \quad x \in \Sigma.$$

Thanks to Corollary 3.10, the integral system (41) admits a solution $\phi \in [L^p(\Sigma)]^n$ if, and only if,

$$\int_\Sigma \psi_i \wedge \omega_i = 0 \tag{43}$$

for any $\psi = (\psi_1, \ldots, \psi_n) \in [L^q_{n-2}(\Sigma)]^n$ such that $R^*\psi = 0$. Arguing as in [13], $R^*\psi = 0$ if, and only if, all the components of ψ are weakly closed (n - 2)-forms. It is clear that (43) is equivalent to conditions (42). □

Lemma 5.2. *For any $f \in [W^{1,p}(\Sigma)]^n$ there exists a solution of the BVP*

$$\begin{cases} w \in \mathcal{S}^p, \\ Ew = 0 \quad \text{in } \Omega, \\ dw = df \quad \text{on } \Sigma. \end{cases} \tag{44}$$

It is given by (3), where the density $\phi \in [L^p(\Sigma)]^n$ solves the singular integral system $R\phi = df$ with R as in (23).

Proof. Consider the following singular integral system:

$$\int_\Sigma d_x[\Gamma(x,y)]\varphi(y)d\sigma_y = df(x), \quad x \in \Sigma, \tag{45}$$

in which the unknown is $\phi \in [L^p(\Sigma)]^n$ and the datum is $df \in [L^p_1(\Sigma)]^n$. In view of Theorem 5.1, there exists a solution ϕ of system (45) because conditions (42) are satisfied. □

In the next result we consider the eigenspace $\mathcal{F}$ of the Fredholm integral system

$$-\frac{1}{2}\psi(x) + \int_\Sigma L_x^{k/(k+2)}[\Gamma(x,y)]\psi(y)d\sigma_y = 0, \quad x \in \Sigma.$$

The dimension of $\mathcal{F}$ is nm. This can be proved as in [[30], p. 63], where the case n = 3 is considered.

Theorem 5.3. *Given $c_0, c_1, \ldots, c_m \in \mathbb{R}^n$, there exists a solution of the BVP*

$$\begin{cases} v \in \mathcal{S}^p, \\ Ev = 0 \quad \text{in } \Omega, \\ v = c_k \quad \text{on } \Sigma_k, k = 0, \ldots, m. \end{cases} \tag{46}$$

It is given by

$$v(x) = \sum_{h=1}^m \sum_{i=1}^n (c_h^i - c_0^i) \int_\Sigma \Gamma(x,y)\Psi_{h,i}(y)d\sigma_y + c_0, \quad x \in \Omega, \tag{47}$$

where $\Psi_{h,i} \in \mathcal{F}(h = 1, \ldots, m,\ i = 1, \ldots, n)$ satisfy the following conditions

$$\int_\Sigma \Gamma(x,y)\Psi_{h,i}(y)d\sigma_y = \delta_{hk}e_i, \quad x \in \overline{\Omega}_k, k = 1, \ldots, m.$$

Proof. Let $\psi_1, \ldots, \psi_{nm}$ be nm linearly independent eigensolutions of the space $\mathcal{F}$. For a fixed $j = 1, \ldots, nm$ we set

$$V_j(x) = \int_\Sigma \Gamma(x,y)\psi_j(y)d\sigma_y, \quad x \in \Omega.$$

Then $L_{-}^{k/(k+2)} V_j = 0$ on Σ. As in [[30], Theorem III, p. 45], this implies that V_j is constant on each connected component of $\mathbb{R}^n \setminus \overline{\Omega}$. Then $V_j = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}_0$ [5] and $V_j(x) = a_j^k$ in Ω_k ($k = 1, ..., m$). For every $k = 1, ..., m$, consider the $n \times nm$ matrix $\mathcal{D}_k$ defined as follows

$$\mathcal{D}_k = \begin{pmatrix} a_{1,1}^k & a_{1,2}^k & \cdots & a_{1,nm}^k \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,1}^k & a_{n,2}^k & \cdots & a_{n,nm}^k \end{pmatrix}.$$

The $nm \times nm$ matrix $\mathcal{D} = (\mathcal{D}_1 \dots \mathcal{D}_m)'$ has a not vanishing determinant. Indeed, if $\det \mathcal{D} = 0$, the linear system $\mathcal{D}\lambda = 0$ admits an eigensolution $\lambda = (\lambda_1, ..., \lambda_{nm}) \in \mathbb{R}^{nm}$. Hence the potential

$$W(x) = \sum_{j=1}^{nm} \lambda_j V_j(x)$$

vanishes not only on $\mathbb{R}^n \setminus \overline{\Omega}_0$, but also on Ω_k ($k = 1, ..., m$). Since this implies $W = 0$ on Σ, we find $W = 0$ in Ω, thanks to the classical uniqueness theorem for the Dirichlet problem. Accordingly, $W = 0$ all over $\mathbb{R}^n$, from which $\sum_{j=1}^{nm} \lambda_j \psi_j \equiv 0$ and this is absurd.

For each $h = 1, ..., m$ and $i = 1, ..., n$, let $(\lambda_{i,1}^h, \dots, \lambda_{i,nm}^h) \in \mathbb{R}^{nm}$ be the solution of the system

$$\sum_{j=1}^{nm} \lambda_{i,j}^h a_j^k = \delta_{hk} e_i, \quad k = 1, \dots, m.$$

Setting

$$\overline{V}_{h,i}(x) = \sum_{j=1}^{nm} \lambda_{i,j}^h V_j(x), \quad x \in \Omega,$$

we get $E\overline{V}_{h,i} = 0$, $\overline{V}_{h,i}|_{\Sigma_0} = 0$ and

$$\overline{V}_{h,i}|_{\Sigma_k} = \sum_{j=1}^{nm} \lambda_{i,j}^h a_j^k = \delta_{hk} e_i, \quad k = 1, \dots, m.$$

Put

$$v(x) = \sum_{h=1}^{m} \sum_{i=1}^{n} (c_h^i - c_0^i) \overline{V}_{h,i}(x) + c_0.$$

The potential v belongs to $\mathcal{S}^p$, thanks to the isomorphism σ introduced in the proof of Lemma 3.12 (for $n = 2$ see Definition 4.6). Moreover

$$v(x)|_{\Sigma_k} = \sum_{h=1}^{m} \sum_{i=1}^{n} (c_h^i - c_0^i) \delta_{hk} e_i + c_0,$$

i.e. $v = ck$ on Σ_k ($k = 0, 1, ..., m$). This shows that v is solution of (46). □

We are now in a position to establish the main result of this section.

Theorem 5.4. *The Dirichlet problem (40) has a unique solution u for every* $f \in [W^{1,p}(\Sigma)]^n$. *If* $n \geq 3$ *or* $n = 2$ *with* Σ_0 *is not exceptional, u is given by (3). If* $n = 2$ *and* Σ_0 *is exceptional, it is given by (39). In any case, the density* ϕ *solves the singular system (45).*

Proof. Let w be a solution of the problem (44). Since $dw = df$ on Σ, $w = f + c_h$ on Σ_h ($h = 0, ..., m$) for some $c_h \in \mathbb{R}^n$. The function $u = w - v$, where v is given by (47), solves the problem (40).

In order to show the uniqueness, suppose that (3) is solution of (40) with $f = 0$. From Corollary 3.10 it follows that the condition $u = 0$ on Σ implies that

$$-\frac{1}{4}\varphi + \left(T^{k/(k+2)}\right)^2 \varphi = 0, \tag{48}$$

where $T^{k/(k+2)}$ is the compact operator given by (26). By bootstrap techniques, (48) implies that ϕ is a Hölder function on Σ. Then u belongs to $[C^{1,\lambda}(\overline{\Omega}) \cap C^2(\Omega)]^n$ and we get that

$$\int_\Omega \mathcal{E}(u, u)dx = 0,$$

from which

$$\mathcal{E}(u, u) = 0 \quad \text{in } \Omega. \tag{49}$$

The solution of (49) is $u(x) = a + Bx$, where $a \in \mathbb{R}^n$ and $B \in \mathscr{S}_n$ are arbitrary. Finally, $u = 0$ in $\bar{\Omega}$ by virtue of the classical uniqueness theorem for the Dirichlet problem. □

Remark 5.5. In order to solve the Dirichlet problem (40), we need to solve the singular integral system (45). We know that this system can be reduced to a Fredholm one by means of the operator $R'^{k/(k+2)}$. This reduction is not an *equivalent reduction* in the usual sense (for this definition see, e.g., [[10], p. 19]), because $\mathcal{N}(R'^{k/(k+2)}) \neq \{0\}$, $\mathcal{N}(R'^{k/(k+2)})$ being the kernel of the operator $R'^{k/(k+2)}$.

However $R'^{k/(k+2)}$ still provides a kind of equivalence. In fact, as in [[31], pp. 253-254], one can prove that $\mathcal{N}(R'^{k/(k+2)}R) = \mathcal{N}(R)$. This implies that *if* ψ *is such that there exists at least a solution of the equation* $R\phi = \psi$, *then* $R\phi = \psi$ *if, and only if,* $R'^{k/(k+2)}R\phi = R'^{k/(k+2)}\psi$.

Since we know that the system $R\phi = df$ is solvable, we have that $R\phi = df$ if, and only if, ϕ is solution of the Fredholm system $R'^{k/(k+2)}R\phi = R'^{k/(k+2)}df$.

Therefore, even if we do not have an equivalent reduction in the usual sense, such Fredholm system is equivalent to the Dirichlet problem (40).

6 The traction problem

The aim of this section is to study the possibility of representing the solution of the traction problem by means of a double layer potential. As we shall see, in an $(m + 1)$-connected domain this is possible if, and only if, the given forces are balanced on each connected component Σ_j of the boundary.

More precisely, we consider the problem

$$\begin{cases} w \in \mathcal{D}^p, \\ Eu = 0 \quad \text{in } \Omega, \\ Lw = f \quad \text{on } \Sigma, \end{cases} \tag{50}$$

where $f \in [L^p(\Sigma)]^n$ is such that

$$\int_\Sigma f(x)(a + Bx)d\sigma_x = 0, \quad a \in \mathbb{R}^n,\ B \in \mathscr{S}_n. \tag{51}$$

We shall prove that, in order to have the existence of a solution of such a problem, condition (51) is not sufficient, but it must be satisfied on each Σ_j, $j = 0, 1, ..., m$ (see Theorem 6.2 below).

If f satisfies the only condition (51), we need to modify the representation of the solution by adding some extra terms (see Theorem 6.4 below).

Lemma 6.1. *Let $w \in \mathcal{D}^2$ be a double layer potential with density $\psi \in [W^{1,2}(\Sigma)]^n$. Then*

$$\int_\Omega \mathcal{E}(w, w)dx = \int_\Sigma wLw\, d\sigma. \tag{52}$$

Proof. Let $(\psi_k)_{k \geq 1}$ be a sequence of functions in $[C^{1,h}(\Sigma)]^n$ $(0 < h < \lambda)$ such that $\psi_k \to \psi$ in $[W^{1,2}(\Sigma)]^n$.

Setting

$$w_k(x) = \int_\Sigma [L_y \Gamma(x, y)]' \psi_k(y) d\sigma_y,$$

we have that $w_k \in [C^{1,h}(\overline{\Omega})]^n$, $Ew_k = 0$ and then

$$\int_\Omega \mathcal{E}(w_k, w_k)dx = \int_\Sigma w_k L w_k d\sigma. \tag{53}$$

From $\psi_k \to \psi$ in $[L^2(\Sigma)]^n$, it follows that $w_k \to w$ in $[L^2(\Sigma)]^n$ because of well-known properties of singular integral operators.

On the other hand we have that $\mathcal{K}_{sj}(d\psi_k) \to \mathcal{K}_{sj}(d\psi)$ in $L^2(\Omega)$. By applying formula (11), we see that $\nabla w_k \to \nabla w$ in $[L^2(\Omega)]^n$. Moreover, since $\mathcal{K}_{sj}(d\psi_k) \to \mathcal{K}_{sj}(d\psi)$ also in $L^2(\Sigma)$, (22) shows that $Lw_k \to Lw$ in $[L^2(\Sigma)]^n$. We get the claim by letting $k \to +\infty$ in (53). □

Theorem 6.2. *Given $f \in [L^p(\Sigma)]^n$, the traction problem (50) admits a solution if, and only if,*

$$\int_{\Sigma_j} f(x)(a + Bx)d\sigma_x = 0 \tag{54}$$

for every $j = 0, 1, ..., m$, $a \in \mathbb{R}^n$ and $B \in \mathscr{S}_n$. The solution is determined up to an additive rigid displacement.

Moreover, (4) is a solution of (50) if, and only if, its density ψ is given by

$$\psi(x) = \int_\Sigma \Gamma(x, y)\phi(y)d\sigma_y, \quad x \in \Sigma, \tag{55}$$

φ being a solution of the singular integral system

$$-\frac{1}{4}\phi + T^2\phi = f, \tag{56}$$

where T is given by (27).

Proof. Assume that conditions (54) hold. If u is the double layer potential with density $\psi \in [W^{1,p}(\Sigma)]^n$, in view of (22) the boundary condition $Lu = f$ turns into the equation

$$R'(d\psi) = f, \tag{57}$$

where R' is given by (24) with $\xi = 1$.

On account of Theorem 5.4, if $n = 2$ and Σ_0 is exceptional, any $\psi \in [W^{1,p}(\Sigma)]^2$ can be written as

$$\int_\Sigma \Gamma(x,y)\phi(y)\, d\sigma_y + c,$$

with $\varphi \in [L^p(\Sigma)]^2$, $c \in \mathbb{R}^2$. In all other cases, ψ can be written as (55) with $\varphi \in [L^p(\Sigma)]^n$. In any case, since $d\psi = R\varphi$ (R being defined by (23)), we infer $R'(d\psi) = R'R\varphi$. Keeping in mind Lemma 3.9, we find that equation (57) is equivalent to (56), with ψ given by (55).

Therefore there exists a solution of the traction problem (50) if, and only if, the singular integral system (56) is solvable.

On the other hand, there exists a solution $\gamma \in [L^p(\Sigma)]^n$ of the singular integral system

$$\frac{1}{2}\gamma + T\gamma = f \tag{58}$$

if, and only if, f is orthogonal to $\mathcal{V}_-$. In view of Lemma 3.12, this occurs if, and only if, (51) is satisfied. Then conditions (54) imply the existence of a solution of (58).

Consider now the singular integral system

$$-\frac{1}{2}\phi + T\phi = \gamma. \tag{59}$$

From Lemma 3.11 the dimension of the kernel $\mathcal{N}(-I/2 + T^*) = \mathcal{V}_+$ is $n(n+1)m/2$ and $\{v_h \chi_{\Sigma_j} : j = 1, \dots, m, h = 1, \dots, n(n+1)/2\}$ is a basis of it. The equation (59) has a solution if, and only if,

$$\int_{\Sigma_j} \gamma v_h d\sigma = 0, \quad j = 1, \dots, m, h = 1, \dots, n(n+1)/2. \tag{60}$$

Since γ is solution of (58), conditions (60) are fulfilled. Indeed, picking $j = 1, \dots, m$ and $h = 1, \dots, n(n+1)/2$, by integrating (58) on Σ_j we find (see (31))

$$\int_{\Sigma_j} f v_h\, d\sigma = \frac{1}{2}\int_{\Sigma_j} \gamma v_h\, d\sigma + \int_{\Sigma_j} v_h(x)\, d\sigma_x \int_\Sigma L_x[\Gamma(x,y)]\gamma(y) d\sigma_y =$$
$$\frac{1}{2}\int_{\Sigma_j} \gamma v_h\, d\sigma + \int_\Sigma \gamma(y)\, d\sigma_y \int_{\Sigma_j} [L_x\Gamma(x,y)]' v_h(x)\, d\sigma_x = \int_{\Sigma_j} \gamma v_h d\sigma.$$

Conditions (60) follow from (54) since the last ones are equivalent to

$$\int_{\Sigma_j} f v_h\, d\sigma = 0, \quad j = 1, \dots, m, h = 1, \dots, n(n+1)/2.$$

Let φ be a solution of (59); taking (58) into account, we have that φ solves (56) and then the traction problem (50) admits a solution.

Conversely, if u is a solution of (50), from Lemma 3.7, we have that

$$\int_{\Sigma_j} f(x)(a+Bx)d\sigma_x = \int_{\Sigma_j} L_+u(x)(a+Bx)d\sigma_x = \int_{\Sigma_j} L_-u(x)(a+Bx)d\sigma_x.$$

By Lemma 6.1, for any fixed $j = 1, \ldots, m$ we have

$$\int_{\Sigma_j} f(x)(a+Bx)d\sigma_x = \int_{\Omega_j} \mathcal{E}(u, a+Bx)dx = 0$$

since $\mathcal{E}(u, a+Bx) = 0$.

Now we pass to discuss the uniqueness. Let u be a solution of (50) with datum $f = 0$. As we know, the condition $L_+u = 0$ is equivalent to the singular integral system $-\frac{1}{4}\phi + T^2\phi = 0$, φ being as in (55), which can be written as

$$\left(-\frac{I}{2}+T\right)\left(\frac{\phi}{2}+T\phi\right) = 0.$$

Set

$$\Xi = \frac{\phi}{2} + T\phi. \tag{61}$$

Since $-\Xi/2+T\Xi = 0$ and the operator $-I/2+T$ can be reduced to Fredholm one, as shown by Kupradze [[1], Chapter IV, §7], Ξ has to be Hölder continuous. By a similar argument, the vector-valued function φ, being solution of the singular integral system (61), is Hölder continuous. Therefore the relevant simple layer potential ψ belongs to $W^{1,2}(\Sigma)$, i.e. $u \in \mathcal{D}^2$. By applying formula (52), we get that u is a rigid displacement in Ω. □

We remark that, by Theorem 6.2, a solution of the traction problem (50) can be written as a double layer potential if, and only if, conditions (54) are satisfied.

In order to consider the problem (50) under the only condition (51), we introduce the following space.

Definition 6.3. *We define $\tilde{\mathcal{D}}^p$as the space of all the functions w written as*

$$w(x) = \int_{\Sigma} [L_y\Gamma(x,y)]'\psi(y)\, d\sigma_y + \sum_{j=1}^{m}\sum_{h=1}^{n(n+1)/2} c_{jh}\int_{\Sigma_j}\Gamma(x,y)\, v_h(y)d\sigma_y, \quad x \in \Omega,$$

where $\psi \in [W^{1,p}(\Sigma)]^n$, $\{v_h : h = 1, \ldots, n(n+1)/2\}$ is an orthonormal basis for $\mathcal{R}$and c_{jh} $\in \mathbb{R}$.

Theorem 6.4. *Given $f \in [L^p(\Sigma)]^n$ satisfying (51), the traction problem*

$$\begin{cases} w \in \tilde{\mathcal{D}}^p, \\ Ew = 0 \qquad \text{in } \Omega, \\ Lw = f \qquad \text{on } \sum \end{cases} \tag{62}$$

admits a solution given by

$$w(x) = \int_{\Sigma} [L_y\Gamma(x,y)]'\psi(y)d\sigma_y + \sum_{j=1}^{m}\sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|}\int_{\Sigma_j} f(t)v_h(t)d\sigma_t \int_{\Sigma_j}\Gamma(x,y)v_h(y)d\sigma_y, \quad x \in \Omega, \tag{63}$$

where $\psi \in [W^{1,p}(\Sigma)]^n$ *is solution of the system*

$$R'(d\psi)(x) = f(x) - \sum_{j=1}^{m} \sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t)\, d\sigma_t \left[\frac{1}{2} v_h(x) \mathcal{X}_{\Sigma_j}(x) + \int_{\Sigma_j} L_x[\Gamma(x,y)] v_h(y) d\sigma_y \right] \text{on } \Sigma. \tag{64}$$

The solution is uniquely determined up to an additive rigid displacement.

Proof. First observe that

$$L_x \left(\int_{\Sigma_j} \Gamma(x,y) v_h(y) d\sigma_y \right) = \begin{cases} \frac{1}{2} v_h(x) + \int_{\Sigma_j} L_x[\Gamma(x,y)] v_h(y) d\sigma_y & x \in \Sigma_j, \\ \int_{\Sigma_j} L_x[\Gamma(x,y)] v_h(y) d\sigma_y & x \in \Sigma \setminus \Sigma_j, \end{cases}$$

for $h = 1, \ldots, n(n+1)/2$ and $j = 1, \ldots, m$. If w is given by (63), taking into account (57), we find that $Lw = f$ if, and only if, is (64) satisfied.

Denote by g the right hand side of (64). In view of Theorem 6.2, $R'(d\psi) = g$ has a solution if, and only if, $\int_{\Sigma_k} g v_l \, d\sigma = 0$ for any $k = 0, 1, \ldots, m$, $l = 1, \ldots, n(n+1)/2$. By integrating on Σ_k ($k = 1, \ldots, m$), for every l we get

$$\begin{aligned}
\int_{\Sigma_k} g(x) v_l(x) d\sigma_x &= \int_{\Sigma_k} f(x) v_l(x) d\sigma_x - \sum_{j=1}^{m} \sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t) d\sigma_t \int_{\Sigma_k} \left[\frac{1}{2} v_h(x) \mathcal{X}_{\Sigma_j}(x) \right. \\
&\quad \left. + \int_{\Sigma_j} L_x[\Gamma(x,y)] v_h(y) d\sigma_y \right] v_l(x) d\sigma_x = \int_{\Sigma_k} f(x) v_l(x) d\sigma_x \\
&\quad - \sum_{j=1}^{m} \sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t) d\sigma_t \int_{\Sigma_k} \frac{1}{2} v_h(x) \mathcal{X}_{\Sigma_j}(x) v_l(x) d\sigma_x \\
&\quad - \sum_{j=1}^{m} \sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t) d\sigma_t \int_{\Sigma_k} v_l(x) d\sigma_x \int_{\Sigma_j} L_x[\Gamma(x,y)] v_h(y) d\sigma_y = \\
&\quad \int_{\Sigma_k} f(x) v_l(x) d\sigma_x - \frac{|\Sigma_k|}{2} \sum_{j=1}^{m} \sum_{h=1}^{n(n+1)/2} \frac{\delta_{jk}\delta_{hl}}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t) d\sigma_t \\
&\quad - \sum_{j=1}^{m} \sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t) d\sigma_t \int_{\Sigma_j} v_h(y) d\sigma_y \int_{\Sigma_k} [L_x \Gamma(x,y)]' v_l(x) d\sigma_x = \\
&\quad \frac{1}{2} \int_{\Sigma_k} f(x) v_l(x) d\sigma_x - \frac{|\Sigma_k|}{2} \sum_{j=1}^{m} \frac{\delta_{jk}\delta_{hl}}{|\Sigma_j|} \int_{\Sigma_j} f(x) v_h(x) d\sigma_x = 0.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_{\Sigma_0} g(x) v_l(x) d\sigma_x &= \int_{\Sigma_0} f(x) v_l(x) d\sigma_x - \sum_{j=1}^{m} \sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t) d\sigma_t \int_{\Sigma_0} \left[\frac{1}{2} v_h(x) \mathcal{X}_{\Sigma_j}(x) \right. \\
&\quad \left. + \int_{\Sigma_j} L_x[\Gamma(x,y)] v_h(y) d\sigma_y \right] v_l(x) d\sigma_x = \int_{\Sigma_0} f(x) v_l(x) d\sigma_x \\
&\quad - \sum_{j=1}^{m} \sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t) d\sigma_t \int_{\Sigma_j} v_h(y) d\sigma_y \int_{\Sigma_0} [L_x \Gamma(x,y)]' v_l(x) d\sigma_x = \\
&\quad \int_{\Sigma_0} f(x) v_l(x) d\sigma_x + \sum_{j=1}^{m} \sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t) d\sigma_t \int_{\Sigma_j} v_l(y) v_h(y) d\sigma_y = \\
&\quad \int_{\Sigma_0} f(x) v_l(x) d\sigma_x + \sum_{j=1}^{m} \int_{\Sigma_j} f(x) v_l(x) d\sigma_x = \int_{\Sigma} f(x) v_l(x) d\sigma_x = 0.
\end{aligned}$$

Finally, assume that w is solution of (62) with $f = 0$. From (63) it follows that $w \in \mathcal{D}^p$ and then w is a rigid displacement in Ω by virtue of the uniqueness proved in Theorem 6.2. □

Endnotes

[1]For the definition of internal (external) angular boundary values see, e.g., [[23], p. 53].

[2]If a simple layer potential u, whose density belongs to $\mathcal{W}_+^0$, is such that $u(x) = c$ in Ω, then $u(x) = c$ in Ω_0. Since $u(\infty) = 0$, we find $u(x) = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}_0$ and this leads to $u = 0$ in $\mathbb{R}^2$, $c = 0$.

[3]It is not difficult to see that $\dot{x}_i\dot{x}_j - (x_i - y_i)(x_j - y_j)|x - y|^{-2} = \mathcal{O}(|y - x|^h), x, y \in \Sigma$.

[4]We remark that for $n \geq 3$ the formula

$$d_x[\Gamma_{ij}(x, y)] = -\frac{1}{\omega_n}\frac{k+2}{2(k+1)}\frac{\delta_{ij}}{2-n}d_x[|x-y|^{2-n}] + \mathcal{O}(|y-x|^{h-n+1})$$

is false.

[5]This is true also for $n = 2$ because $\int_\Sigma \psi_j \, d\sigma = 0$.

Authors' contributions

The authors declare that the work was realized in collaboration with same responsibility. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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